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Convex Functions & Optimization

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Abstract - My research paper is based on the recent work in interior-point methods, specifically those methods that keep track of both the primal and dual optimization variables (hence primal-dual methods). These methods are special because they are numerically stable under a wide range of conditions, so they should work well for many different types of constrained optimization problems.However, you can always find a constrained optimization problem that is difficult enough to break these methods.

Keywords - Introduction, Types of Optimization, Graphical Minima, Convex function, Convex vs. Non-convex, Functions, Convex Hull, Test for convexity and Concavity, Convex Region, Solving Techniques, Some common convex OP's, LP Visualization, Quadratic Programming, QP Visualization, Interior Point Method,CVX:Convex Optimization, Building Convex Functions, Verifying Convexity Remarks , References

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1.Introduction

Optimization is the mathematical discipline which is concerned with finding the maxima and minima of functions, possibly subject to constraints. It helps in various field such as Architecture, Nutrition, Electrical circuits, Economics , Transportation,etc.

2.Types of Optimization

a)A real function of n variables

$$f(x_1, x_2, \ldots, x_n)$$

with or without constrains

$$\min\ f(x,y) = x^2 + 2y^2$$

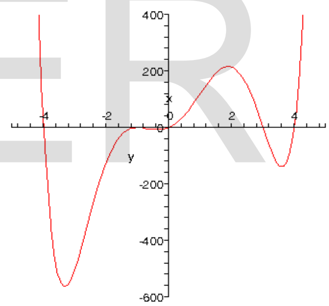
b)Unconstrained optimization

c) Optimization with constraints

$$\min\ f(x,y) = x^2 + 2y^2$$
$$x > 0$$
or
$$\min\ f(x,y) = x^2 + 2y^2$$
$$-2 < x < 5, y \geq 1$$
or
$$\min\ f(x,y) = x^2 + 2y^2$$
$$x + y = 2$$

3.Graphical Minima

a)To find the minimum of the function



What is special about a local max or a local min of a function **f** (x)?
at local max or local min **f**'(x)=0
f"(x) > 0 if local min
f"(x) < 0 if local max

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4.Convex Function

a)Definition

The weighted mean of function evaluated at any two points is greater than or equal to the function evaluated at the weighted mean of the two points

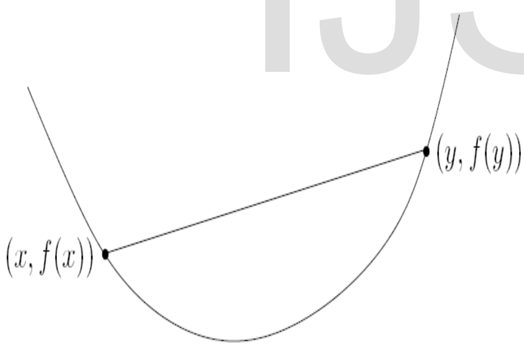
$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

$$\text{if } \alpha + \beta = 1,\ \alpha \geq 0,\ \beta \geq 0$$

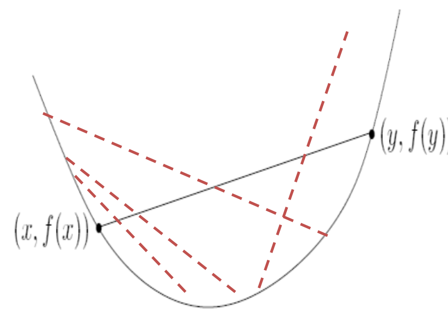
b)Procedure

a)Pick any two points x, y and evaluate along the function, f(x), f(y)
b)Draw the line passing through the two points f(x) and f(y)
c)Convex if function evaluated on any point along the line between x and y is below the line between f(x) and f(y)

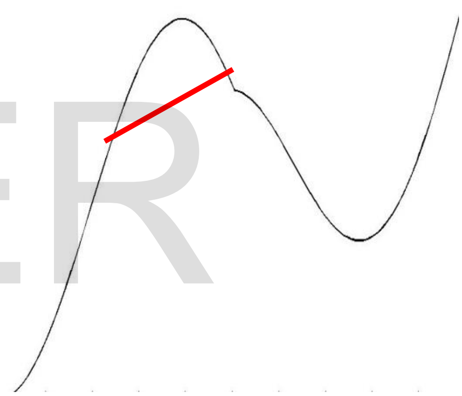
c)Graph



d)Examples



Convex



Not Convex

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5.Local Optima is Global (simple proof)

proof: suppose x is locally optimal and y is optimal with $f_0(y) < f_0(x)$

x locally optimal means there is an $R > 0$ such that

$$z \text{ feasible}, \quad \|z - x\|_2 \leq R \quad \Longrightarrow \quad f_0(z) \geq f_0(x)$$

consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$

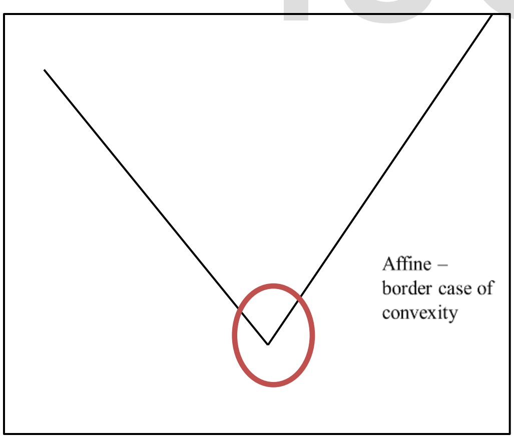
- $\|y - x\|_2 > R$, so $0 < \theta < 1/2$
- z is a convex combination of two feasible points, hence also feasible
- $\|z - x\|_2 = R/2$ and

$$f_0(z) \leq \theta f_0(x) + (1 - \theta) f_0(y) < f_0(x)$$

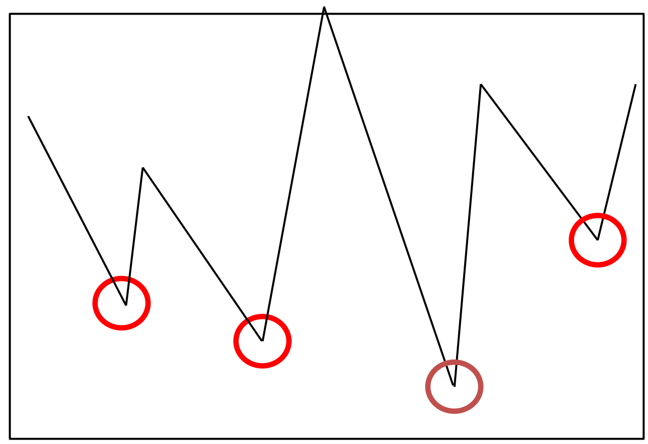
which contradicts our assumption that x is locally optimal

6. Convex vs. Non-convex

Convex



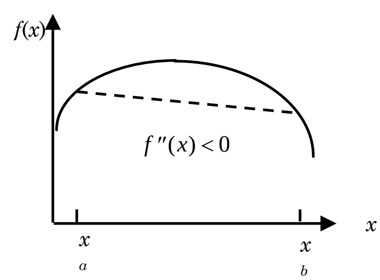
Not Convex



7. Functions

Convex

A function is called convex (strictly convex) if $\succeq$ is replaced by $\preceq$(<).

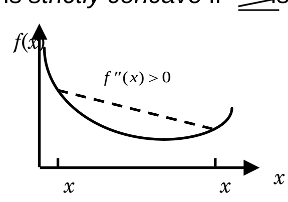


Concave

A function is called *concave* over a given region *R* if:

$$f(\theta \mathbf{x}_a + (1-\theta)\mathbf{x}_b) \geq \theta f(\mathbf{x}_a) + (1-\theta) f(\mathbf{x}_b)$$
$$\text{where:} \quad \mathbf{x}_a, \mathbf{x}_b \in R, \quad \text{and } 0 \leq \theta \leq 1.$$

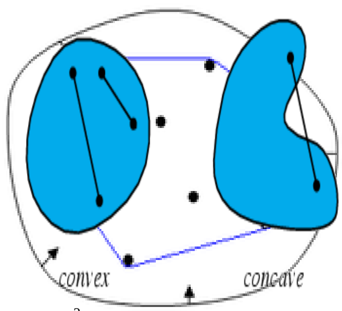
The function is *strictly concave* if $\succeq$ is replaced by >.



8.Convex Hull

A set C is **convex** if every point on the line segment connecting *x* and *y* is in *C.*

The **convex hull** for a set of points X is the minimal convex set containing X.



If $f''(x) = \frac{\partial^2 f}{\partial x^2} \leq 0$ then $f(x)$ is concave.

If $f''(x) = \frac{\partial^2 f}{\partial x^2} \geq 0$ then $f(x)$ is convex.

For a *multivariate function* *f*(**x**) the conditions are:-

f(**x**)	**h(x) Hessian Matrix**
Strictly convex	+ve def
convex	+ve semi def
concave	-ve semi def
strictly concave	-ve def

9.Test for Convexity and Concavity

H is -ve def (-ve semi def) iff

$$\mathbf{x}^T\mathbf{H}\mathbf{x} > 0 \ \ (\geq 0), \ \ \forall \mathbf{x} \neq \mathbf{0}.$$

$$\mathbf{x}^T\mathbf{H}\mathbf{x} < 0 \ \ (\leq 0), \ \ \forall \mathbf{x} \neq \mathbf{0}.$$

<u>*Convenient tests*</u>: **H**(**x**) is strictly convex (+ve def) (convex) (+ve semi def)) if:

1. all eigenvalues of **H**(**x**) are $> 0 \ (\geq 0)$

or 2. all principal determinants of **H**(**x**) are $> 0 \ \ (\geq 0)$

Example:

$$f(x) = 2x_1^2 - 3x_1x_2 + 2x_2^2$$

$$\frac{\partial f(\mathbf{x})}{\partial x_1} = 4x_1 - 3x_2 \quad \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} = 4 \quad \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} = -3$$

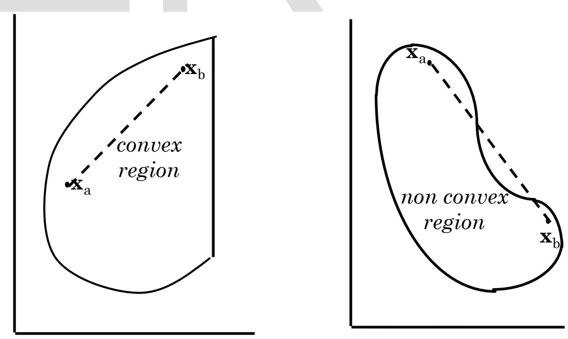
$$\frac{\partial f(\mathbf{x})}{\partial x_2} = -3x_1 + 4x_2 \quad \frac{\partial^2 f(\mathbf{x})}{\partial x_2^2} = 4$$

$$\therefore \mathbf{H}(\mathbf{x}) = \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}, \quad \Delta_1 = 4, \quad \Delta_2 = \begin{vmatrix} 4 & -3 \\ -3 & 4 \end{vmatrix} = 7$$

$$\text{eigenvalues: } |\lambda \mathbf{I}_2 - \mathbf{H}| = \begin{vmatrix} \lambda - 4 & 3 \\ 3 & \lambda - 4 \end{vmatrix} = \lambda^2 - 8\lambda + 7 = 0$$

$\Rightarrow \lambda_1 = 1, \ \lambda_2 = 7.$ Hence, $f(\mathbf{x})$ is strictly convex.

10.Convex Region



A convex set of points exist if for any two points, $\mathbf{x}_a$ and $\mathbf{x}_b$, in a region, all points:

$$\mathbf{x} = \mu \mathbf{x}_a + (1 - \mu)\mathbf{x}_b, \quad 0 \leq \mu \leq 1$$

on the straight line joining $\mathbf{x}_a$ and $\mathbf{x}_b$ are in the set.
If a region is completely bounded by concave functions then the functions form a convex region.

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11.Solving Techniques

Can use definition (prove holds) to prove
If function restricted to any line is convex, function is convex
If 2X differentiable, show hessian >= 0
Often easier to:
Convert to a known convex OP
E.g. QP, LP, SOCP, SDP, often of a more general form
Combine known convex functions (building blocks) using operations that preserve convexity
Similar idea to building kernels

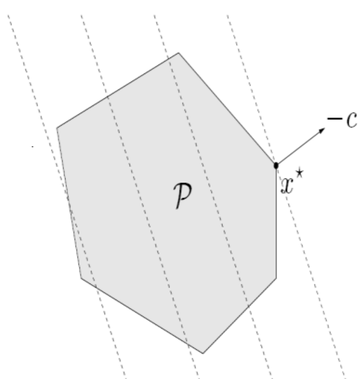
12. Some common convex OPs

Of particular interest for this book and chapter:
linear programming (LP) and *quadratic programming (QP)*
LP: affine objective function, affine constraints

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

-e.g. LP SVM, portfolio management

13.LP Visualization



Note: constraints form *feasible set*
-for LP, polyhedra

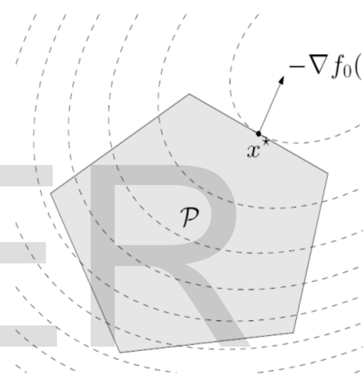
14.Quadratic Program

➨QP: Quadratic objective, affine constraints

➨LP is special case
➨Many SVM problems result in QP, regression
➨If constraint functions quadratic, then Quadratically Constrained Quadratic Program (QCQP)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

15.QP Visualization



16.Interior Point Method

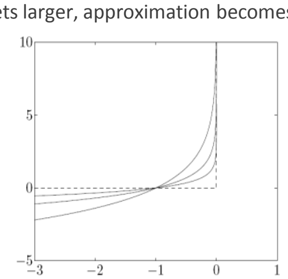
➨Solve a series of equality constrained problems with Newton's method
➨Approximate constraints with log-barrier (approx. of indicator)

$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^m I_-(f_i(x)) \\ \text{subject to} & Ax = b, \end{array}$$

$$I_-(u) = \begin{cases} 0 & u \le 0 \\ \infty & u > 0 \end{cases}$$

$$\begin{array}{ll} \text{minimize} & f_0(x) - (1/t)\sum_{i=1}^m \log(-f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

As t gets larger, approximation becomes better



17.CVX: Convex Optimization

a)Introduction

CVX is a Matlab toolbox
Allows you to flexibly express convex optimization problems
Translates these to a general form and uses efficient solver (SOCP, SDP, or a series of these)

All you have to do is design the convex optimization problem
Plug into CVX, a first version of algorithm implemented
More specialized solver may be necessary for some applications

b)CVX - Examples

I)

Quadratic program: given H, f, A, and b

```
cvx_begin
variable  x(n)
            minimize (x'*H*x + f'*x)
subject to
             A*x  >= b
cvx_end
```

II)

SVM-type formulation with L1 norm

```
cvx_begin
  variable w(p)
  variable b(1)
  variable e(n)
  expression by(n)
  by = train_label.*b;
  minimize( w'*(L + I)*w + C*sum(e)  +
l1_lambda*norm(w,1) )
  subject to
                  X*w + by >= a - e;
                  e >= ec;
cvx_end
```

18.Building Convex Functions

From simple convex functions to complex: some *operations that preserve complexity*
Nonnegative weighted sum
Composition with affine function
Pointwise maximum and supremum
Composition
Minimization
Perspective   (g(x,t) = tf(x/t))

19.Verifying Convexity Remarks

➨For more detail and expansion, consult the referenced text, *Convex Optimization*
➨Geometric Programs also convex, can be handled with a series of SDPs (skipped details here)
➨CVX converts the problem either to SOCP or SDM (or a series of) and uses efficient solver

20.References

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